

Övningstentamen 2

You should always try to include all your workings out and remember to end up with a result that directly answers the question. If you find any mistakes, please let me know at rjjo@KTH.se

Problem 1 The matrix have a real double-root, so the general set of solutions is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} C + At \\ D + Bt \end{pmatrix}$$

There is no exponential function present, since $e^{mt} = e^0 = 1$. To find A and B , note that $x'_1 = y'_1 = 0$, so

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3A - B \\ 9A - 3B \end{pmatrix}$$

By setting $A = 1$, we get $B = 3$. Using these values, we have $x'_2 = A = 1$ and $y'_2 = B = 3$, so

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3C - D \\ 9C - 3D \end{pmatrix}$$

Choosing $C = 1$ gives $D = 2$. Thus, the general solution is

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1+t \\ 2+3t \end{pmatrix}$$

Problem 2 Whenever we use reduction of order, we set $\frac{dy}{dx} = p$, and when we want to use y as the independent variable, we set $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dy}{dx} \frac{dp}{dy} = p \frac{dp}{dy}$. Plugging this into the equation gives

$$y^2 p \frac{dp}{dy} + p^3 = 0$$

which can be separated to

$$\frac{1}{p^2} dp = -\frac{1}{y^2} dy$$

giving $-\frac{1}{p} = \frac{1}{y} + c_1$. Using $p = \frac{dy}{dx}$, we can separate again

$$\left(\frac{1}{y} + c_1 \right) dy = -dx$$

which gives $\ln y + c_1 y = -x + c_2$ or $ye^{c_1 y} = m_2 e^{-x}$ as an implicit solution.

Problem 3 Set $y_1 := \sum_{n=0}^{\infty} a_n x^n$ and differentiate termwise

$$y_1'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

and multiply termwise

$$xy_1 = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Now that the powers of x in the two series match, we need to remove the first term of y_1'' (since it starts at $n = 0$). But that is easily done, by pulling it out of the series:

$$y_1'' - xy_1 = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 2a_2 + \sum_{n=1}^{\infty} ((n+2)(n+1)a_{n+2} - a_{n-1}) x^n = 0$$

Since the above should specifically hold at $x = 0$, we can read off $a_2 = 0$. The recursive relation is

$$a_{n+2} = \frac{1}{(n+1)(n+2)} a_{n-1} \quad \text{for} \quad n \geq 1$$

The first two are given by the initial conditions: $a_0 = y_1(0) = 1$ and $a_1 = y_1'(0) = 0$. Applying the recursive relation gives

$$a_3 = \frac{1}{6} a_0 = \frac{1}{6} \quad a_4 = \frac{1}{12} a_1 = 0 \quad a_5 = \frac{1}{20} a_2 = 0 \quad a_6 = \frac{1}{30} a_3 = \frac{1}{180}$$

Problem 4 To determine the critical points of the system, we consider the first equation $f(x, y) = -2xy$. This is 0 iff $x = 0$, $y = 0$ or both. The second equation, $g(x, y) = y - x + xy - y^3$, satisfies

$$g(0, y) = y - y^3 = y(1 - y^2) \quad g(x, 0) = -x$$

The first relation shows that, if $x = 0$, then $g = 0$ for $y = 0$ and $y = \pm 1$. The second relation shows that, if $y = 0$, then the only possibility for x is to be 0 as well. We have three critical points: $(0, 0)$, $(0, 1)$ and $(0, -1)$. The Jacobian of the system is

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -2y & -2x \\ y-1 & 1+x-3y^2 \end{pmatrix}$$

at the three critical points:

$$(0, 0) : \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \quad (0, 1) : \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad (0, -1) : \begin{pmatrix} 2 & 0 \\ -2 & -2 \end{pmatrix}$$

At $(0, 0)$ the matrix has eigenvalues 0 and 1, which implies instability, however, because one eigenvalue is 0, it does not fall under any of the major cases either. We can therefore not say anything about the behaviour of the original system at this point. At $(0, 1)$ the matrix has a double eigenvalue -2 , this is not a major case, so we can not conclude anything about the type of critical point. But, since the eigenvalues are both negative, we can conclude that the point is asymptotically stable, which implies asymptotic stability for the original system as well. The last point falls under a major case: It is a saddle point for both the linearized and original systems.

Problem 5 To find the orthogonal trajectories, we need to first find the differential equation for the family

$$(x - c)^2 + y^2 = c^2 \quad \text{or} \quad x^2 - 2xc + y^2 = 0$$

Differentiating both sides with respect to x gives

$$2x - 2c + 2yy' = 0 \quad \text{or} \quad y' = \frac{c - x}{y}$$

using the right hand version of the original equation, we isolate $c = \frac{x^2 + y^2}{2x}$, which we use to eliminate it from the differential equation

$$y' = \frac{x^2 + y^2}{2xy} - \frac{x}{y} = \frac{y^2 - x^2}{2xy}$$

The differential equation for the orthogonal trajectories is then the negative reciprocal of this equation.

$$y' = -\frac{2xy}{y^2 - x^2} \quad \text{or} \quad 2xy + (y^2 - x^2)y' = 0$$

We are given the hint of an integrating factor that only depends on y , so the formula is (for $M = 2xy$ and $N = y^2 - x^2$)

$$\frac{\partial N / \partial x - \partial M / \partial y}{M} = \frac{-2x - 2x}{2xy} = -\frac{2}{y}$$

which gives the integrating factor

$$\mu(y) = e^{\int -2/y \, dy} = \frac{1}{y^2}$$

and the exact equation is then

$$\frac{1}{y^2} \left(2xy \, dx + (y^2 - x^2) \, dy \right) = \frac{2x}{y} \, dx + \left(1 - \frac{x^2}{y^2} \right) \, dy$$

Integrating, to solve the exact equation,

$$\int \frac{2x}{y} \, dx + \int \left(1 - \frac{x^2}{y^2} \right) \, dy - \int \frac{\partial}{\partial y} \left(\int \frac{2x}{y} \, dx \right) \, dy = \frac{x^2}{y} + y$$

and the family of orthogonal trajectories is

$$\frac{x^2}{y} + y = c \quad \text{or} \quad x^2 + (y - m)^2 = m^2$$

where the second form has $m = \frac{c}{2}$ to showcase that it is very similar to the original family of curves.

Problem 6 Set (as in the description)

$$\begin{pmatrix} x \\ y \end{pmatrix} = v_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

and differentiate

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= v_1 \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} + v_2 \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix} + v'_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v'_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} v_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \end{pmatrix} + v'_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v'_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + v'_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v'_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \end{aligned}$$

where we have used the fact that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ satisfies the homogenous system. The above shows that we need the relation

$$v'_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v'_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

for the non-homogenous system to be satisfied. The matrix of part (b) has eigenvalues 0 and 1, so the general solution to the homogenous system is

$$c_1 \begin{pmatrix} A \\ B \end{pmatrix} + c_2 \begin{pmatrix} Ce^t \\ De^t \end{pmatrix}$$

By plugging the two solutions into the system, we can choose $A = 1$ which gives $B = 1$ and $C = 3$ which gives $D = 2$. The relation that v'_1 and v'_2 has to satisfy, is then

$$v'_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + v'_2 \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix} = \begin{pmatrix} 4 \\ t \end{pmatrix}$$

which gives $v'_1 = 3t - 8$ and $v'_2 = (4 - t)e^{-t}$. By integration $v_1 = \frac{3}{2}t^2 - 8t$ and $v_2 = (t - 3)e^{-t}$. The solution to the non-homogenous system is

$$\left(\frac{3}{2}t^2 - 8t \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (t - 3)e^{-t} \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix} = \begin{pmatrix} \frac{3}{2}t^2 - 5t - 9 \\ \frac{3}{2}t^2 - 6t - 6 \end{pmatrix}$$