

Övningstentamen 1

Disclaimer: This is a very rough set of solutions as i wanted it to fit on two pages. You should always try to include all your workings out and remember to end up with a result that directly answers the question. If you find any mistakes, please let me know at rjjo@KTH.se

Problem 1 Set $f(x, y) := \frac{2xy}{x^2 - y^2}$, then we find the solution for the substitution $z = \frac{y}{x}$

$$\int \frac{1}{f(1, z) - z} dz = \int \frac{1 - z^2}{z + z^3} dz = \ln \left(\frac{z}{1 + z^2} \right)$$

This should be equal to

$$\int \frac{1}{x} dx = \ln(x) + c$$

So

$$\frac{z}{1 + z^2} = mx$$

giving the implicit solution $m(y^2 + x^2) - y = 0$.

Problem 2 The usual pair of linearly independent solutions are

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} A_1 \cos 2t - A_2 \sin 2t \\ B_1 \cos 2t - B_2 \sin 2t \end{pmatrix} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_1 \sin 2t + A_2 \cos 2t \\ B_1 \sin 2t + B_2 \cos 2t \end{pmatrix}$$

from the relation $x'_1 = 2x_1 + 8y_1$, we get

$$-2A_1 \sin 2t - 2A_2 \cos 2t = 2A_1 \cos 2t - 2A_2 \sin 2t + 8B_1 \cos 2t - 8B_2 \sin 2t$$

which corresponds to

$$\begin{cases} -2A_1 = -2A_2 - 8B_2 \\ -2A_2 = 2A_1 + 8B_1 \end{cases}$$

choosing $A_1 = B_1 = 2$ gives $A_2 = -10$ and $B_2 = 3$. The specific solution to the initial value problem

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} x_1(0) \\ y_1(0) \end{pmatrix} + c_2 \begin{pmatrix} x_2(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2c_1 - 10c_2 \\ 2c_1 + 3c_2 \end{pmatrix}$$

giving $c_1 = -\frac{2}{13}$ and $c_2 = -\frac{3}{13}$. Remember to write out the solution!

Problem 3 Assume $y_2(x) = xv(x)$ and plug this into the equation, to obtain

$$x^3 v'' + x^3 v' = 0$$

Solving the equation $u' + u = 0$ gives $u = e^{-x}$. Since $u = v'$, we have that

$$y_2(x) = -xe^{-x}$$

The general solution is

$$y_g(x) = c_1 x + c_2 x e^{-x}$$

Problem 4 Remember to first reduce it to the usual form of a second order linear equation (factor of y'' should be 1):

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = \frac{1}{\sqrt{x}}$$

variation of parameters leads to

$$\begin{cases} v_1' \frac{\cos(x)}{\sqrt{x}} + v_2' \frac{\sin(x)}{\sqrt{x}} = 0 \\ v_1' \left(-\frac{\cos(x)}{2x^{3/2}} - \frac{\sin(x)}{\sqrt{x}} \right) + v_2' \left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{3/2}} \right) = \frac{1}{\sqrt{x}} \end{cases}$$

Using the first equality, reduces the second to

$$-v_1' \frac{\sin(x)}{\sqrt{x}} + v_2' \frac{\cos(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

which is clearly seen to have solutions $v_1' = -\sin(x)$ and $v_2' = \cos(x)$, which gives $v_1 = \cos(x)$ and $v_2 = \sin(x)$. It follows that

$$y_p = \frac{\cos^2(x)}{\sqrt{x}} + \frac{\sin^2(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Problem 5 As always, assume the Lyapunov function is on the form $E(x, y) = ax^{2m} + by^{2n}$ (for $a, b > 0$) and calculate

$$\begin{aligned} \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G &= 2max^{2m-1}(y^3 - x^3) + 2nby^{2n-1}(-2xy^2 - 4x^2y - 2y^3) \\ &= -2max^{2m+2} - 4nby^{2n}(2x^2 + y^2) + 2max^{2m-1}y^3 - 4nbxy^{2n+1} \end{aligned}$$

The first two terms are always negative definite (since they contain only even powers of x and y), but we need the latter two to annihilate:

$$2max^{2m-1}y^3 - 4nbxy^{2n+1} = 0$$

which is the case, if $m = n = 1$, $a = 2$ and $b = 1$. Our Lyapunov function is $E(x, y) = 2x^2 + y^2$.

Problem 6 Assume a solution of Frobenius type: $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m}$, for some m . Then

$$xy_1'' = \sum_{n=0}^{\infty} (m+n)(n+m-1)a_n x^{n+m-1} = m(m-1)a_0 x^{m-1} + \sum_{n=0}^{\infty} (m+n+1)(n+m)a_{n+1} x^{n+m}$$

Since $xy_1'' + y = 0$ and y does not contain terms of order $m-1$, we must have

$$m(m-1)a_0 x^{m-1} = 0$$

in particular: $m = 0$ or $m = 1$. The larger of the two will always give a formal solution, so we take $m = 1$ (and consider $m = 0$ at the end), in which case

$$xy_1'' + y = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+1} + a_n] x^{n+1}$$

this gives the recursive relation

$$a_{n+1} = -\frac{a_n}{(n+2)(n+1)}$$

and, setting $a_0 = 1$, we get

$$a_1 = -\frac{1}{2}, \quad a_2 = -\frac{a_1}{6} = \frac{1}{12}, \quad a_3 = -\frac{a_2}{12} = -\frac{1}{144}$$

Using the ratio test,

$$\left| \frac{a_{n+1} x^{n+m+1}}{a_n x^{n+m}} \right| = \frac{1}{(n+2)(n+1)} |x| \rightarrow 0$$

and, since the limit is < 1 for all x , we conclude that the series converges for all x . A priori, $m = 0$ may still give a viable Frobenius-type solution, to see if this is the case we need to check whether the recursive formula for $a_{m_1-m_2} = a_1$ ($m_1 = 1$ and $m_2 = 0$) gives a sensible answer for $m = 0$:

$$m(m+1)a_1 + a_0 = 0$$

This is not the case, since $a_0 \neq 0$ (as it is always assumed to be). There can therefore not exist a second linearly independent Frobenius-type solution.